Pressure Loss in Unsteady Annular Channel Flow

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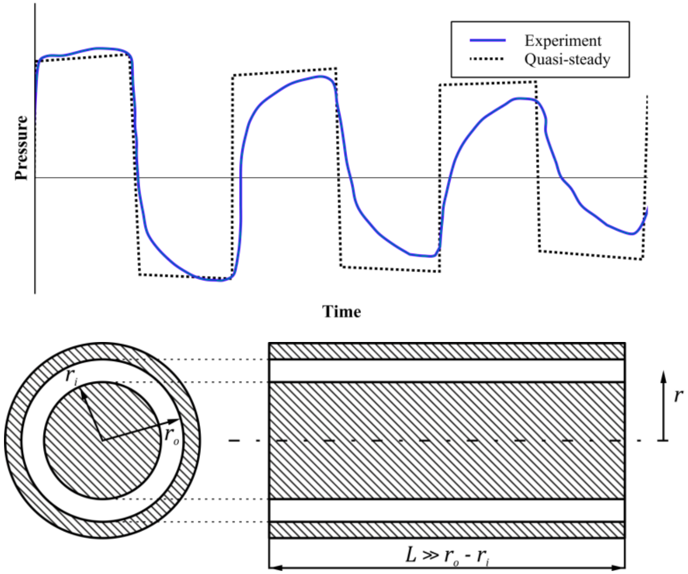

Outline

1	Introduction to the problem of unsteady pressure loss
2	Pressure loss in unsteady annular channel flow
3	Plane channel approximation
4	Summary & Outlook

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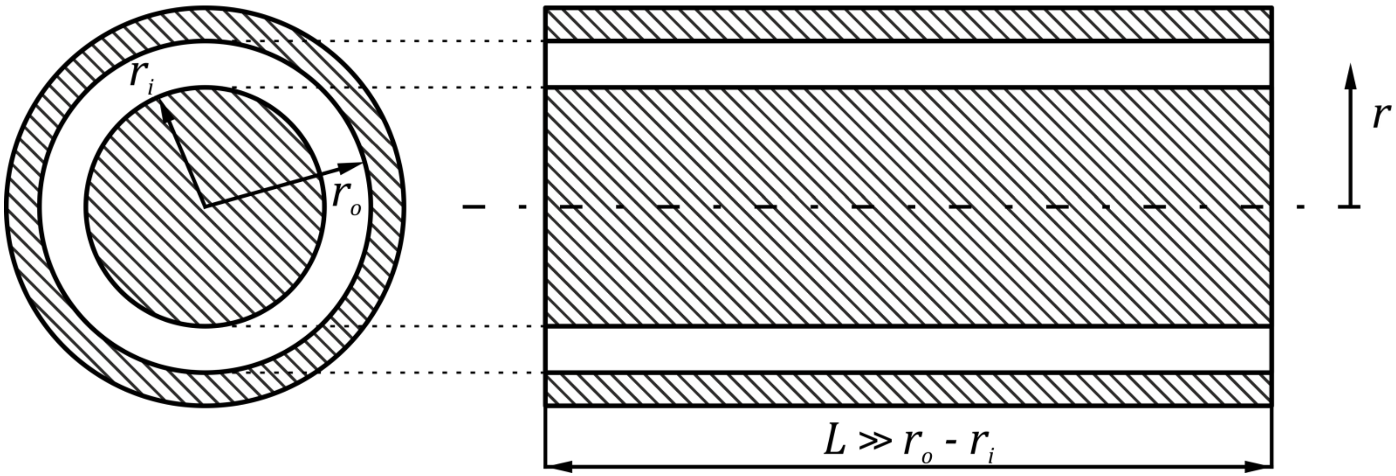
Introduction to the problem of unsteady pressure loss

- What is an annular channel?
- An annular channel is created by mounting a cylinder within a pipe:



- Key design parameter: radius ratio $\varrho = \frac{r_i}{r_o}$
- Radius ratio ranging from $0 < \varrho \leq 1$, depending on application
- $\varrho = 0$: circular pipe without internal cylinder
- $\varrho \to 1$: vanishing gap height $h = r_o - r_i$, typically found in sealing gaps

Introduction to the problem of unsteady pressure loss

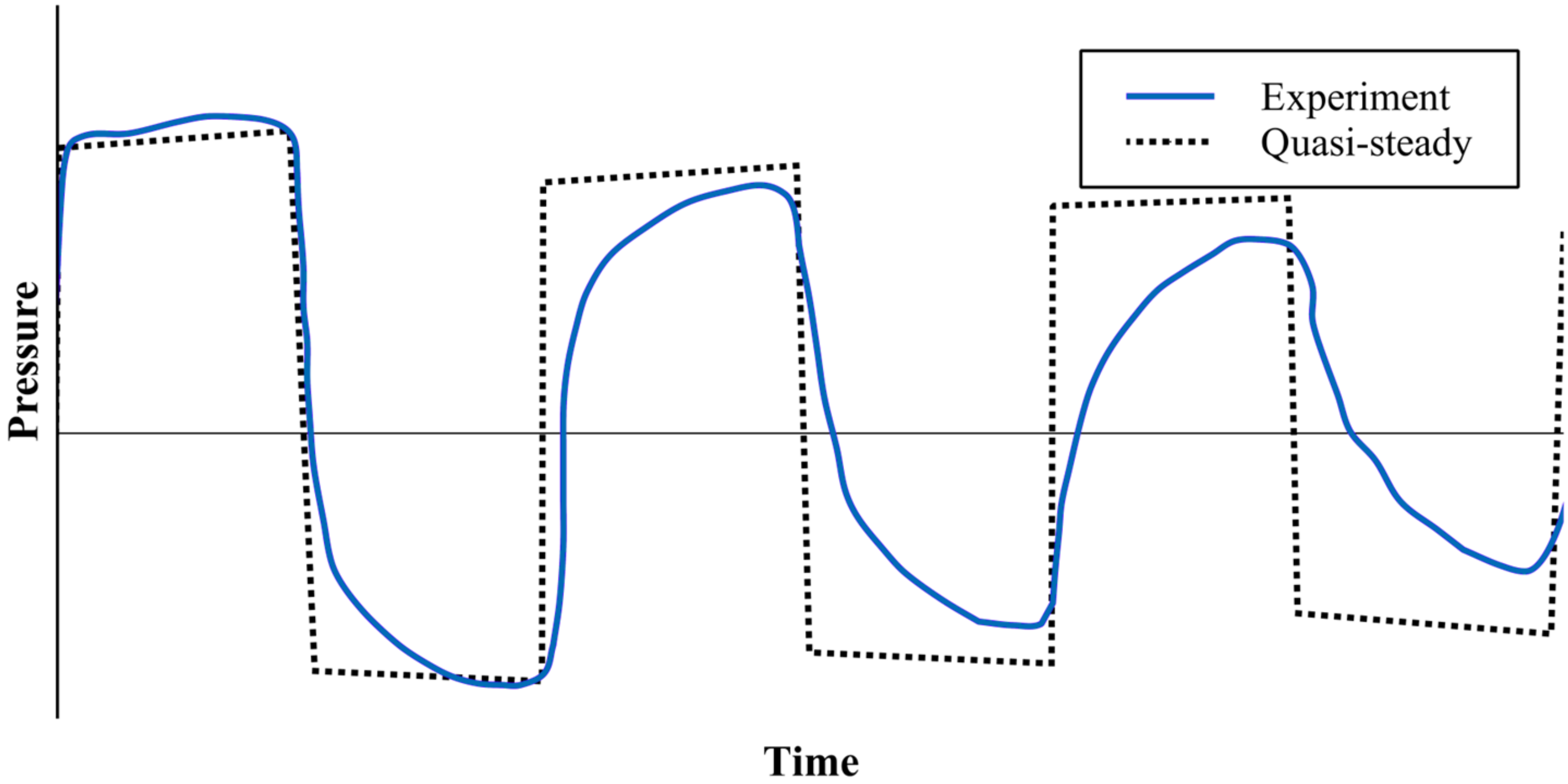
- Hydraulic engineering problem: calculation of pressure loss Δp for a given area-averaged flow velocity $\bar{u}$ or flow rate Q (also vice versa)
- CFD simulations are too time-consuming for design studies → 1D simulation
- For **steady** laminar flow, an analytical expression for $\Delta p(\bar{u})$ is known:

$$\frac{\Delta p}{\Delta z} = \frac{\eta}{r_o^2} \frac{8}{1 + \varrho^2 + \frac{(1 - \varrho^2)}{\ln \varrho}} \bar{u}$$

- For **unsteady** laminar flow, **no analytical expression is known** yet
- First guess: Take the **instantaneous** value of $\bar{u}(t)$ and calculate the unsteady pressure loss based on the formula above
- This method is referred to as the **quasi steady approach**
- Quasi steady approach gives **exact** results for unsteady flows with relatively **low frequencies**
- Quasi steady approach **fails** to predict pressure loss in highly dynamic unsteady flows

Introduction to the problem of unsteady pressure loss

- Example for highly dynamic event: **Water hammer** experiment



- How can unsteady pressure loss be calculated?

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Pressure loss in unsteady annular channel flow

- Pressure loss is proportional to **wall shear stress**

$$\frac{\Delta p}{\Delta z} \propto \tau$$

- Wall shear stress is proportional to radial **velocity gradient**

$$\tau \propto \eta \frac{\partial u}{\partial r}$$

- First step: Determination of velocity profile based on Navier-Stokes (NS) equation
- NS equation (axial direction) in cylindrical coordinates:

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

- Strategy:
  - Perform **Laplace transform** of NS equation
  - Apply boundary conditions (no slip between fluid and cylinder/pipe walls)
  - Solve for u^*

$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = su^*$$

$$u^*(r_i) = u^*(r_o) = 0$$

Pressure loss in unsteady annular channel flow

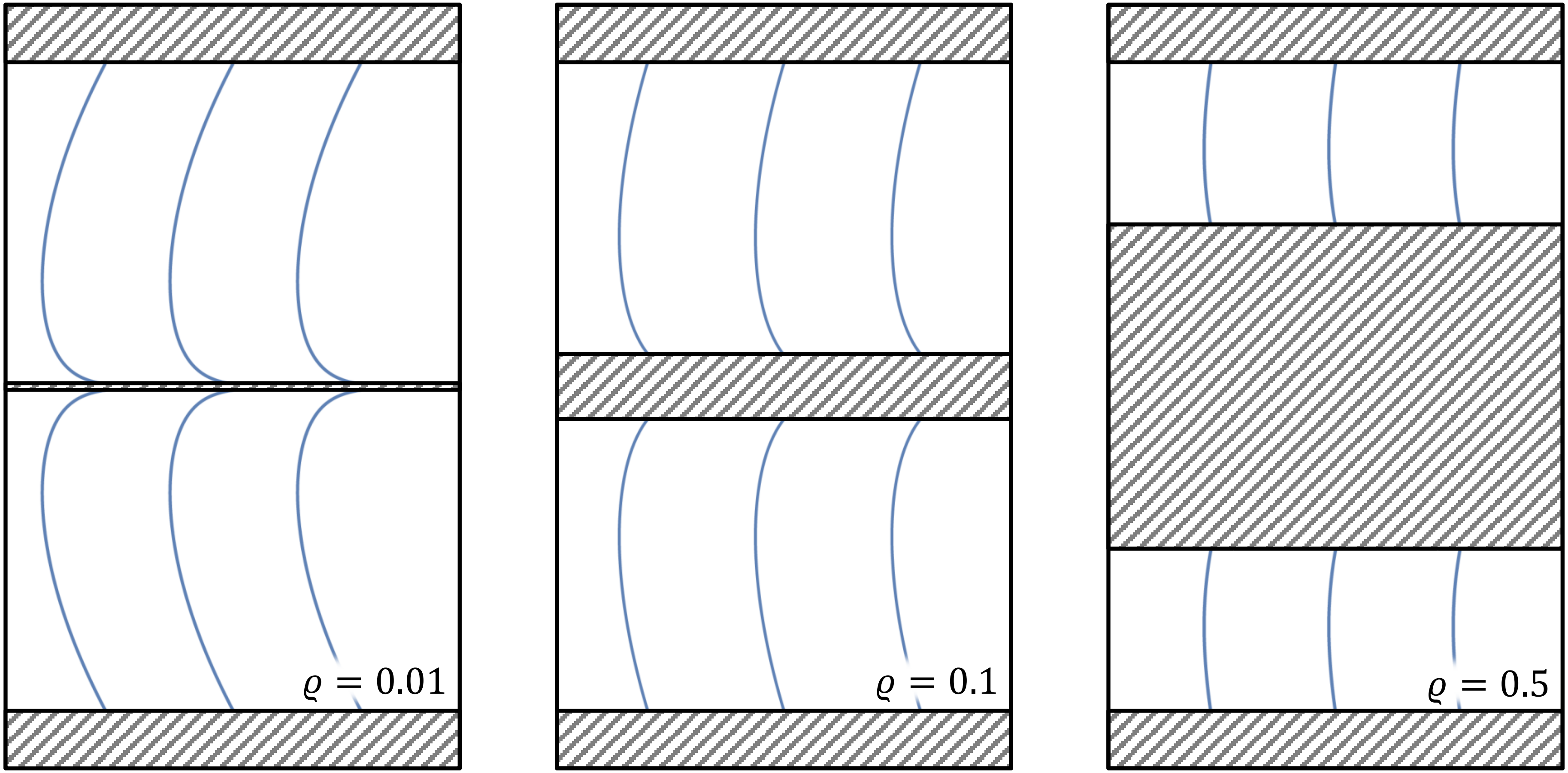
- Result: Laplace transform of radial velocity profile u^* as a function of transformed axial pressure gradient $\frac{\partial p^*}{\partial z}$:

$$u^*(r) = \frac{1}{s\rho}\frac{\partial p^*}{\partial z}\left\{\frac{I_0\left(r\sqrt{\frac{s}{\nu}}\right)\left[K_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - K_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right] - K_0\left(r\sqrt{\frac{s}{\nu}}\right)\left[I_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - I_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right]}{I_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)K_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - I_0\left(r_o\sqrt{\frac{s}{\nu}}\right)K_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)} - 1\right\}$$

- For oscillating flow (oscillation frequency f), the normalized velocity profile depends on two non-dimensional parameters:
  - Gap Womersley number $Wo_h = h\sqrt{\frac{|s|}{\nu}} = h\sqrt{\frac{2\pi f}{\nu}}$
  - Radius ratio ϱ
- How do changes in gap Womersley number and radius ratio influence the velocity profile?

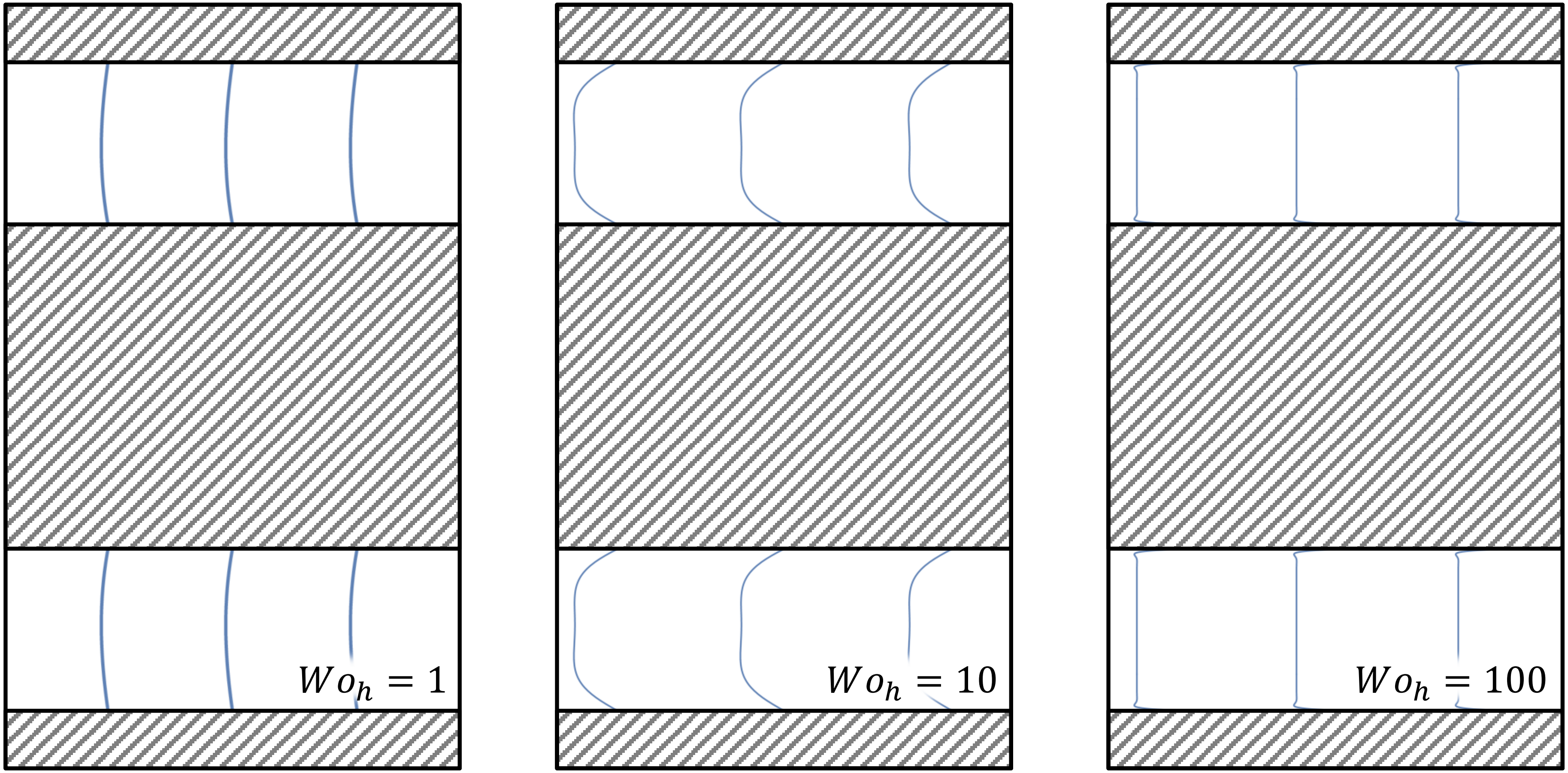
Pressure loss in unsteady annular channel flow

- Influence of radius ratio ϱ on velocity profile ($Wo_h = 1$)



Pressure loss in unsteady annular channel flow

- Influence of gap Womersley number Wo_h on velocity profile ($\varrho = 0.5$)



Pressure loss in unsteady annular channel flow

- Pressure loss can be derived based on velocity profile
- Laplace transform of unsteady pressure loss:

$$\frac{\Delta p^*}{\Delta z} = \frac{\eta}{r_o^2} F^*\left(r_o\sqrt{\frac{s}{\nu}}, \varrho\right) \bar{u}^*$$

$$F^* = \frac{2}{\dfrac{(1-\varrho^2)\left[I_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)K_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - I_0\left(r_o\sqrt{\frac{s}{\nu}}\right)K_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right] r_o\sqrt{\frac{s}{\nu}}}{\left[I_1\left(r_o\sqrt{\frac{s}{\nu}}\right) - \varrho I_1\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right]\left[K_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - K_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right] + \left[I_0\left(r_o\sqrt{\frac{s}{\nu}}\right) - I_0\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right]\left[K_1\left(r_o\sqrt{\frac{s}{\nu}}\right) - \varrho K_1\left(\varrho r_o\sqrt{\frac{s}{\nu}}\right)\right]} - 2}$$

- In oscillating flow, pressure loss depends on gap Womersley number and radius ratio
- For flows with arbitrary temporal distribution: derivation of inverse Laplace transform of pressure loss formula

Pressure loss in unsteady annular channel flow

- Inverse Laplace transform of a product of two functions leads to a convolution integral:

$$\mathcal{L}^{-1}\{F^*\bar{u}^*\} = \int_0^t \bar{u}(t_1)F(t-t_1)\,\mathrm{d}t$$

- Problem: There is no inverse Laplace transform of F^*, since F^* does not converge to zero for $s \to \infty$
- Solution: Expansion with s/s:

$$F^*\bar{u}^* = \frac{F^*}{s}s\bar{u}^* = W^*s\bar{u}^* = W^*\mathcal{L}\left\{\frac{\partial\bar{u}}{\partial t}\right\}(s)$$

- Inverse Laplace transform of W^* can be derived by employing residue theorem from complex analysis
- Problems:
  - Different inverse Laplace transform for every radius ratio ϱ
  - Determination of inverse Laplace transform cumbersome due to numerical behaviour of Bessel functions
- Solution?

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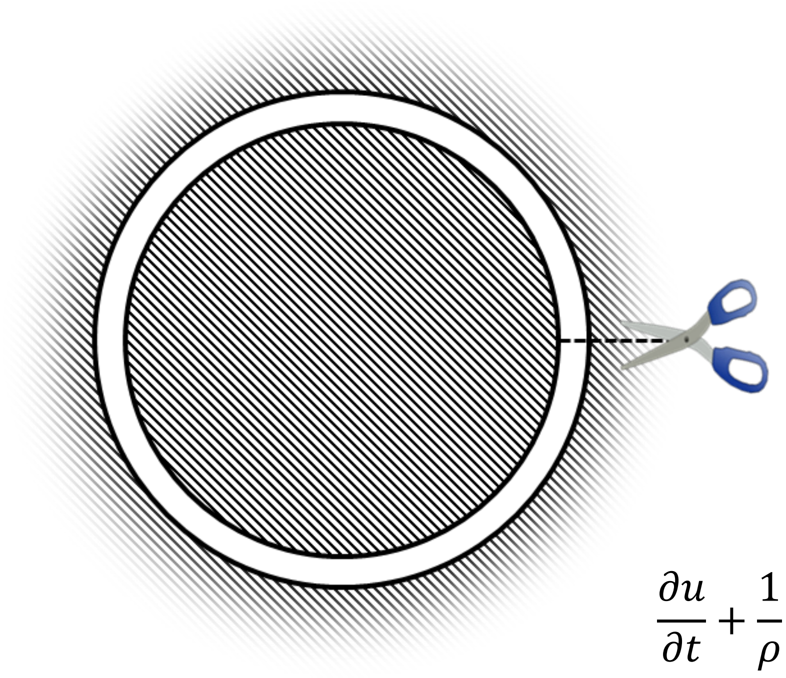
Plane channel approximation

- Analysis of velocity profile: with decreasing channel height ($\varrho \to 1$), the velocity profile approaches that of a plane channel
- Solution for small channel heights: replace actual problem of annular channel flow with plane channel approximation



$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right)$$

Plane channel approximation

- Approximated time domain expression for pressure loss in unsteady annular channel flow:

$$\frac{\Delta p}{\Delta z} = \frac{12\eta}{h^2}\bar{u}(t) + \frac{\eta}{h^2}\int_0^t \frac{\partial \bar{u}}{\partial t}(t_1) W_d(t - t_1)\,\mathrm{d}t$$

- For $t > 0.0023\,h^2/\nu$:

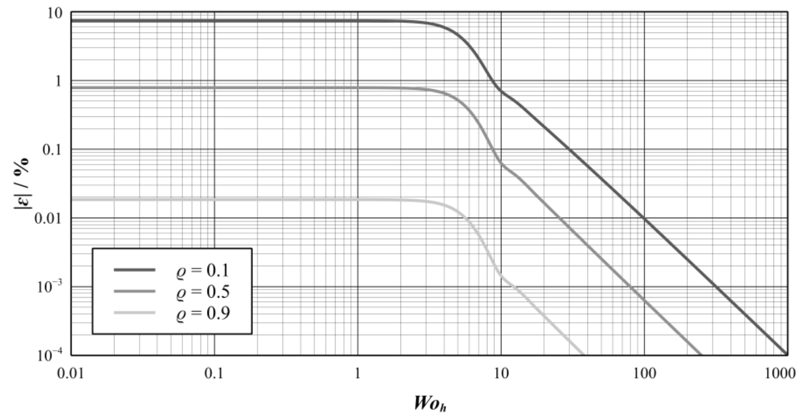
$$W_d(t_n) = 8\left(e^{-80.76 t_n} + e^{-238.72 t_n} + e^{-475.59 t_n} + e^{-791.43 t_n} + \cdots\right)$$

$$t_n = \frac{t\nu}{h^2}$$

- For $t < 0.0023\,h^2/\nu$:

$$W_d(t_n) = \frac{2}{\sqrt{\pi t_n}} - 8 + \frac{16}{\sqrt{\pi}}\sqrt{t_n} + 16 t_n + \frac{128}{3\sqrt{\pi}}\sqrt{t_n^3} + 32 t_n^2 + \cdots$$

- Error analysis: approximation error ε less than 1% for radius ratios down to $\varrho = 0.5$
- Approximation suitable for most practical purposes!



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Summary & Outlook

- Summary:
  - The velocity profile for unsteady annular channel flow was derived
  - Velocity profile depends on radius ratio & gap Womersley number
  - Pressure loss depends on radius ratio & gap Womersley number
  - Inverse Laplace transform of pressure loss formula is cumbersome due to numerical behaviour of Bessel functions and variations in radius ratio
  - For sufficiently narrow channels: plane channel approximation
  - Approximation error of less than 1 % for radius ratios larger than 0.5

- Outlook:
  - Derivation of a curve fit of the approximated weighting function for efficient simulation of transient events in annular channel flow
  - Implementation into the simulation software DSHplus
  - Expansion to shear flow (moving pipe wall or moving cylinder)

Thank you for your attention!

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